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LETTER TO THE EDITOR

QCD heat kernel in covariant gauge

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Abstract. We report the calculation of the fourth coefficient in an expansion of the heat kernel of a non-minimal, non-Abelian kinetic operator in an arbitrary background gauge in arbitrary spacetime dimension. The fourth coefficient is shown to bring a non-trivial gauge dependence due to the contribution of the lowest order off-shell gauge-invariant structure.

In this letter we continue a programme of studying the quasiclassical expansion of the heat kernel of the gluon kinetic operator considered in the arbitrary spacetime dimension and arbitrary covariant gauge started in [1]. The importance of studying the heat kernel and its quasiclassical expansion is due to the important role it plays in quantum field theory and mathematical physics. In quantum field theory a heat kernel is a unique device for computing the Green functions and the one-loop quantum corrections to the classical action of the theory (effective action) in the case of an inhomogeneous background field. This, in turn, helps to perform an efficient calculation of charge renormalization and anomalies [2]. The method also provides a possibility of performing an infinite resummation of the quasiclassical expansion and the corresponding contributions to the effective action [3]. In mathematical physics the heat kernel is a central object in the spectral geometry of manifolds. Its quasiclassical expansion generates the invariants of the manifold with respect to the symmetry transformations of fields and gauge connections defined on it (see e.g. [4]).

Physically, the heat kernel is a way of describing the propagation of the eigenmodes of the system under consideration. As is well known, the particular feature needed to determine a non-Abelian theory is the necessity of getting rid of the unphysical degrees of freedom by fixing a gauge. This leads to a gauge-dependent description of the propagation of the physical degrees of freedom, and, correspondingly, to a gauge-dependent heat kernel $K(s)$

$$K(s) = \exp(-\mathcal{W}s) \tag{1}$$

where $\mathcal{W}$ is an inverse propagator of a non-Abelian gauge boson (which is actually a covariant Laplacian on the manifold characterized by the external gauge connections, see below) and s is the proper time. In the following we shall consider the case of massless gauge bosons (QCD gluons). The non-trivial application of the heat kernel arises when one considers a propagation of the quantum gluons on the manifold characterized by external gauge connections (external non-Abelian fields). The trace of a heat kernel then depends on the invariants constructed from these external fields and the question to be studied is the interplay between the gauge dependence of a heat kernel and various structures (invariants) appearing in its expansion. As it follows from the results presented below, the resulting structure is non-trivial.

Calculational method

We begin by introducing the basic notation. To do this, let us consider a second-order elliptic operator $\mathcal{W}$ on the 2ω-dimensional manifold $\mathcal{M}$. By definition the heat kernel operator corresponding to $\mathcal{W}$ is obtained by its exponentiation as given in (1). We shall be interested in the expansion of the trace of the matrix elements of the heat kernel taken at one spacetime point. For the second-order operator this expansion has the form (see, e.g., [2]):

$$\mathrm{Tr}\langle x|K(s)|x\rangle = \sum_{k=0}^{\infty} b_{-\omega+k}(\mathcal{W}, \omega \mid x) s^{-\omega+k} \tag{2}$$

where the coefficients $b_{-\omega+k}$ are the so-called Seeley coefficients. These coefficients are the invariants of the manifold $\mathcal{M}$ and the trace of the heat kernel can be considered as a generating function for these invariants.

For the specific case of a non-Abelian gauge theory the kinetic operator for gluons propagating in the external non-Abelian field and considered in the arbitrary covariant background gauge has the form

$$\mathcal{W}^{ab}_{\mu\nu} = -D^2(A)^{ab}\delta_{\mu\nu} - 2f^{acb}G^c_{\mu\nu} - \left(\frac{1}{\alpha} - 1\right) D^{ac}_\mu D^{cb}_\nu \tag{3}$$

where α is a gauge-fixing parameter, f^{abc} are the structure constants of the corresponding Lie algebra, D^a_μ is a covariant derivative containing the external field potential A_μ and $G_{\mu\nu}$ is the corresponding field strength. The Seeley coefficients are thus, generally speaking, also the functions of a gauge parameter

$$\mathrm{Tr}\langle x|\mathrm{e}^{-\mathcal{W}^{ab}_{\mu\nu}s}|x\rangle = \sum_{k=0}^{\infty} b_{-\omega+k}(G_{\mu\nu}, \omega, \alpha \mid x) s^{-\omega+k} \tag{4}$$

and are invariant with respect to the gauge transformations of the external field potentials A^a_μ. To calculate the functional trace in (2) we shall use the basis of plane waves (see, e.g., [5, 6]):

$$\mathrm{Tr}\langle x|\mathrm{e}^{-\mathcal{W}s}|x\rangle = \mathrm{Tr}\int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \mathrm{e}^{-\mathrm{i}px}\langle x|\mathrm{e}^{-\mathcal{W}s}|x\rangle \mathrm{e}^{\mathrm{i}px}$$

where the trace is performed over the Lorentz and colour indices. The $\exp[\mathrm{i}px]$ should be pushed through the operator to the left then cancelled by $\exp[-\mathrm{i}px]$. This has the effect that all differentiation operators in $\mathcal{W}$ becoming shifted: $\partial_\mu \to \partial_\mu + \mathrm{i}p_\mu$. Thus we have

$$\mathrm{Tr}\,\mathrm{e}^{-\mathcal{W}s} = \mathrm{Tr}\int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \mathrm{e}^{-s\mathcal{W}(\partial_\mu \to \partial_\mu + \mathrm{i}p_\mu)} 1 \tag{5}$$

where the operator in the right-hand side of (5) acts on 1. For calculational purposes it is convenient to separate this operator into the parts having zero, first and second order in the external field correspondingly:

$$\mathcal{W}(\partial_\mu \to \partial_\mu + \mathrm{i}p_\mu) = \mathcal{W}_0 - \mathrm{i}\mathcal{W}_1 - \mathcal{W}_2$$

where

$$\begin{aligned}
&\mathcal{W}_0^{\nu\nu} = p^2\left(P_\perp^{\mu\nu} + \frac{1}{\alpha}P_\|^{\mu\nu}\right) \\
&P_\perp^{\mu\nu} = \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \qquad P^{\mu\nu} = \frac{p^\mu p^\nu}{p^2} \\
&\mathcal{W}_1^{\mu\nu} = 2p_\alpha D_\alpha \delta^{\mu\nu} + \beta(p^\mu D^\nu + p^\nu D^\mu) \\
&\mathcal{W}_2^{\mu\nu} = D^2\delta^{\mu\nu} + 2G^{\mu\nu} + \beta D^\mu D^\nu \qquad \beta \equiv \frac{1}{\alpha} - 1\,.
\end{aligned} \tag{6}$$

In the above expression we have suppressed the colour indices, which could be trivially restored, for example, $G_{\mu\nu} \to G_{\mu\nu}^{ab} = f^{acb}G_{\mu\nu}^c$.

To obtain the expansion of this operator in s we use ordinary perturbation theory:

$$\begin{aligned}
\mathrm{Tr}\,\mathrm{e}^{(-\mathcal{W}_0+\mathrm{i}\mathcal{W}_1+\mathcal{W}_2)s} &= \mathrm{Tr}\,K_0(s) + \mathrm{Tr}(K_0(s)(\mathrm{i}\mathcal{W}_1 + \mathcal{W}_2)) + \int_0^s \mathrm{d}s_1(s-s_1) \\
&\times \mathrm{Tr}(K_0(s-s_1)(\mathrm{i}\mathcal{W}_1 + \mathcal{W}_2)K_0(s_1)(\mathrm{i}\mathcal{W}_1 + \mathcal{W}s_2)) + \cdots
\end{aligned} \tag{7}$$

where $K_0(s)$ is the free propagator

$$K_0(s) = \mathrm{e}^{-\mathcal{W}_0^{\mu\nu}s} = \mathrm{e}^{-sp^2}(P_\perp^{\mu\nu} + \mathrm{e}^{-s\beta p^2}P_\|^{\mu\nu})\,. \tag{8}$$

The expressions for the Seeley coefficients are obtained by collecting the terms of a given order in covariant derivatives.

Next we shall apply the above described method to calculate the fourth Seeley coefficient for the gluon kinetic operator in the background α-gauge.

Fourth Seeley coefficient in α-gauge

We start with recalling the first three Seeley coefficients [1]. They have the following form:

$$b_{-\omega} = \frac{N_c^2 - 1}{2^{2\omega}\pi^\omega}(2\omega - [1-\alpha^\omega]) \tag{9}$$

$$b_{-\omega+1} = \frac{N_c}{2^{2\omega}\pi^\omega}(2\omega - [1-\alpha^{\omega-1}])\frac{\Gamma(\omega)-\Gamma(\omega+1)}{\Gamma(\omega)/\omega}A_\mu^a A_\mu^a \equiv 0 \tag{10}$$

$$b_{-\omega+2} = \frac{N_c}{2^{2\omega}\pi^\omega}(2\omega - [1-\alpha^{\omega-2}])G_{\mu\nu}^a G_{\mu\nu}^a\,. \tag{11}$$

The fourth Seeley coefficient corresponds to collecting the contributions of the sixth order in the covariant derivatives. In this order there exist two gauge-invariant structures. These are

$$G_3 \equiv f^{abc}G_{\mu\nu}^a G_{\nu\rho}^b G_{\rho\mu}^c \qquad I_3 \equiv (D_\mu^{ab}G_{\mu\nu}^b)(D_\rho^{ac}G_{\rho\nu}^c)\,.$$

The second invariant I_3 is non-zero only for the fields that do not obey the classical equations of motion, i.e. off-shell. The expression for the fourth Seeley coefficient $b_{-\omega+3}$ reads

$$b_{-\omega+3} = -\frac{N_c}{2^{2\omega}\pi^\omega}\left[\tfrac{1}{180}\left(2\omega - \left[1-\alpha^{\omega-3}\right]\right)(G_3 - I_3) + \tfrac{2}{3}I_3 + \xi(\omega,\alpha)I_3\right] \tag{12}$$

where

$$\begin{aligned}
\xi(\omega,\alpha) = \frac{1}{12\omega(\omega-1)(\omega-2)}\Bigg\{&10\omega^2 - 25\omega + 6 - 6\frac{\alpha}{1-\alpha}(2\omega-1) + \frac{\alpha^{\omega-2}}{1-\alpha}[2\omega(\omega-1) \\
&+\omega(11-2\omega)\alpha + 3(\omega-2)\alpha^2]\Bigg\}\,.
\end{aligned} \tag{13}$$

This expression is the main result of our letter. We see that it is indeed gauge invariant. The most interesting property of the above expression is its dependence on the gauge fixing for quantum gluons, i.e. its α dependence. First of all, one immediately sees that in the limit $\alpha \to 1$ (Feynman gauge) the function $\xi(\omega, 1)$ equals zero identically for any ω and for $\omega = 2$ the remaining part of b_1 coincides with that obtained in [6] in this particular case. Let us note that function $\xi(\omega, \alpha)$ also does not have poles at any ω and the corresponding limits are

$$\xi(0, \alpha) = -\frac{1}{24}\left\{\frac{2}{\alpha^2} - \frac{9}{\alpha} + 13 + 6\frac{\ln(\alpha)}{(1-\alpha)}\right\} \tag{14}$$

$$\xi(1, \alpha) = -\frac{1}{12}\left\{2\left(\frac{1}{\alpha} + 2\right) + 3\frac{\ln(\alpha)}{(1-\alpha)}(3-\alpha)\right\} \tag{15}$$

$$\xi(2, \alpha) = \frac{1}{12}\left\{3(7-\alpha) + 2\frac{\ln(\alpha)}{(1-\alpha)}(2+7\alpha)\right\}. \tag{16}$$

From a physical point of view the expressions (14) and (15) do not make any sense but they show that 'explicit' poles in (13) are not the actual poles. In fact, the function $\xi(\omega, \alpha)$ reflects a non-trivial off-shell contribution to $b_{-\omega+3}$. Its dependence on α and ω is complicated and at this moment we do not have any further comments on its structure.

The second important property of the coefficient $b_{-\omega+3}$ is that the on-shell contribution, which is proportional to G_3 is proportional to the factor $2\omega - [1 - \alpha^{\omega-3}]$. A comparison with the expressions for other coefficients (equations (9)–(11)) shows that the on-shell contribution to the heat kernel in order n contains a universal factor $2\omega - [1 - \alpha^{\omega-n}]$. A very plausible conjecture is that this is valid for all terms in the heat kernel expansion. As was shown in [1], this corresponds to the appearance of the contributions proportional to the $\ln \alpha$ in the ζ-regularized gluon contribution of the effective action.

Conclusion

Following the line of research described in our previous paper [1] we have calculated the fourth coefficient in the quasiclassical expansion for the trace of a heat kernel operator for the kinetic operator for a non-Abelian gauge boson in an arbitrary background gauge and arbitrary spacetime dimension. The new coefficient is of the sixth order in the covariant derivatives with respect to the background field gauge-potentials and is the first one containing two different invariants with respect to gauge transformations of the background field. One of the invariants is an on-shell one ($G_3 \equiv f^{abc}G^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu}$), and another one ($I_3 \equiv (D^{ab}_\mu D^b_{\mu\nu})D^{ac}_\rho G^c_{\rho\nu})$) is an off-shell one, i.e. it vanishes for the external fields obeying classical equations of motion. The structure of the coefficient corresponding to the on-shell contribution is proved to follow the pattern seen in the previously calculated coefficients (see [1]). The off-shell contribution is found to be a complicated function of the quantum gauge fixing parameter and the spacetime dimension.

The results obtained in this letter are of considerable importance for the analysis of the general structure of the contributions to the effective action. In particular, it is important to understand the structure of the heat kernel expansion and the corresponding contributions to the effective action at finite temperature (i.e. on the cylindrical spacetime manifold).

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